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An Exact Inflationary Solution to Non-Minimally Coupling

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Abstract We find a new exact inflationary solution to non-minimally coupled scalar field from a specific $H(\varphi)$. The inflation is driven by the evolution of the scalar field with a new inflation potential. The spectral index of the scalar density fluctuations n_s is consistent with the result of WMAP3 for the power-law flat ΛCDM model. Our solution relaxes the constraint to the quartic coupling constant, e.g. when $\xi = 10^3$, $\lambda \leq 8.9 \times 10^{-11}$.

Keywords Exact solution · Non-minimally coupling · Spectral index

1 Introduction

As we know, inflation theory is one of the most successful cosmological theories, it gives us not only the solution of major cosmological puzzles such as the horizon and flatness problems, but also the generation of primordial perturbations. However, the development of inflation theory is not smoothly. The first semi-realistic model of inflation was proposed by A. Starobinsky in 1979 [1]. It was based on the investigation of a conformal anomaly in quantum gravity and on the assumption of a homogeneous and isotropic universe at the very beginning. Obviously, the assumption somewhat diverged the original goals of proposing inflation scenario. More achievements of gravitational anomaly see [2, 3]. On the other hand, the "old inflation" [4] and the "new inflation" [5–8] represented a substantial but incomplete modification of the big bang theory. They both implied that the scalar field φ takes only a specific value at the initial time, and they still assumed that the universe was in a state of thermal equilibrium from the very beginning that it was relatively homogeneous and large

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enough to survive until the beginning of inflation, which both seem somewhat artificial. Fortunately, the chaotic inflation scenario, proposed by A. Linde in 1983 [9], perfectly avoids these drawbacks. According to this scenario, a more natural ideal is the scalar field φ can take a sort of values at the very beginning, and inflation may begin even if there was no thermal equilibrium in the early universe. Moreover, it has little limit to potential: chaotic inflation can occur in any theory where the potential has a sufficiently flat region which allows the existence of the slow-roll regime. The chaotic inflation scenario has been accepted most widely by now.

The minimal coupling case of chaotic inflation is based on the assumption that the coupling parameter in the action describing the interaction between the gravitational field and the scalar field is zero (i.e. $\xi = 0$). Recently, the non-minimal coupling case (i.e. $\xi \neq 0$), which is the subject of this paper, has attracted a great deal of attention as a possible improvement of the minimal coupling [10]. The researches on quantum field in curved spacetime show that the non-minimal coupling is actually required when the space-time curvature is large [11].

Most detailed studies of inflation were made by using numerical integration, or by employing some approximation schemes. We know that numerical integration is not an ideal method of theoretical study but only an expedient, and any approximation scheme has its own limitation. For example, the slow-roll approximation [8], the most widely used one, which neglects the most slowly changing terms in the equations of motion and works well in many minimal coupling cases. But it must eventually be failed because the inflation has to end. Moreover, even weak violations from the inflation would result in large deviations from the observable predications such as the spectrum of density perturbations. As to nonminimal case, it is more difficult and unreliable to achieve this approximation. Fortunately, a new way out of slow-roll has been found. Generally, the Hubble parameter H cannot be an exact constant, but can vary along the scalar field φ. A convenient approach is to express H directly as a function of φ instead of as a function of time t, namely $H = H(\varphi)$ [12]. This method is creditable because it allows the full dynamical behavior of field to be investigated in terms of the function $H(\varphi)$ thus does not have to assume that friction terms in the field equations dominate or that the field kinetic energy is negligible. This approach can be used to solve some exact inflationary solutions, see [11–15] for example.

In this paper we will make use of the new method to get an exact inflationary solution in the chaotic model with non-minimally coupling. The paper is organized as follows: In Sect. 2 we present an exact inflationary solution to non-minimally coupling. In Sect. 3 we calculate the spectral index of density fluctuations and the quartic coupling constant and check them with the observed values. The last section is our discussion.

2 Exact Inflationary Solution

In the chaotic inflationary scenario we are interested in the evolution of the domains of the universe with initially sufficiently large φ ($\varphi \geq m_{Pl}$, where m_{Pl} is the Planck mass). And it is assumed that the scalar field dominates the evolution of the universe and that no forms of matter other than the scalar field are included in the Lagrangian density. Let us start from the action of non-minimally coupling [16]

$$S = \int d^4x\sqrt{-g}\left[\left(\frac{1}{16\pi G} + \frac{1}{2}\xi\varphi^2\right)R + \frac{1}{2}g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} - V(\varphi)\right], \tag{1}$$

where G is the Newton gravitational constant, R is the scalar curvature, $V(\varphi)$ is the generic potential for the scalar field φ, and $\xi\varphi^2 R/2$ is the coupled term between the scalar field and the space-time curvature. It is clearly that the action will decay to minimally coupling one when $\xi = 0$. In this paper we assume a homogeneous distribution for the scalar field, and the metric is taken to be that of a Robertson-Walker universe

$$ds^2 = dt^2 - a^2(t)dx^2, \tag{2}$$

where $a(t)$ is the scale factor, and the spatial curvature is set to be zero since it is unimportant in this context.

From functions (1) and (2) we can get the dynamic equations

$$\ddot{\varphi} + 3H\dot{\varphi} - 6\xi\varphi(\dot{H} + 2H^2) + V'(\varphi) = 0, \tag{3}$$

$$3H^2\left(\frac{1}{8\pi G} + \xi\varphi^2\right) = \left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi) - 6\xi H\varphi\dot{\varphi}\right], \tag{4}$$

$$\dot{H}\left(\frac{1}{8\pi G} + \xi\varphi^2\right) = -\left(\frac{1}{2} + \xi\right)\dot{\varphi}^2 + \xi H\varphi\dot{\varphi} - \xi\varphi\ddot{\varphi}, \tag{5}$$

where $V'(\varphi) \equiv dV(\varphi)/d\varphi$, and there are only two independent equations according to the Bianchi identity. From the above three equations we can further obtain the following functions [17],

$$V(\varphi) = \left[\frac{3}{8\pi G} + \frac{3}{2k}\left(1 + \frac{3}{k}\right)\varphi^2\right]H^2 - \frac{1}{2}\Phi^2(\varphi), \tag{6}$$

$$\dot{\varphi} = \frac{3}{k}\varphi H \pm \Phi(\varphi), \tag{7}$$

$$\Phi(\varphi) = C(\varphi)\varphi^{-(k+1)}, \tag{8}$$

$$C'(\varphi) = \mp\varphi^{k+1}\left\{\left[\frac{k}{4\pi G\varphi} + \left(1 + \frac{3}{k}\right)\varphi\right]H' + 2\left(1 + \frac{3}{k}\right)H\right\}, \tag{9}$$

where $k \equiv 1/(2\xi)$, the primes denote the differentiation with respect to φ, and when $H' > 0$ choose the up signs of $\pm$ and $\mp$, otherwise, use the down ones. From the above four functions we see that once $H(\varphi)$ is defined, $C(\varphi)$ can be solved from (9), and thus $\Phi(\varphi)$ is determinate according to (8), then (7) and (6) can be solved. In view of this, we won't start our work from $V(\varphi)$ as usually do, but from a specific $H(\varphi)$ to get an exact inflationary solution of the universe.

In this paper we set

$$H(\varphi) = -\alpha\varphi + \beta, \tag{10}$$

where α and β are positive constants. From (10) and (9), we find

$$C = -\frac{3}{k}\alpha\varphi^{k+3} + \frac{2(k+3)}{k(k+2)}\beta\varphi^{k+2} - \frac{k}{4\pi G(k+1)}\alpha\varphi^{k+1}, \tag{11}$$

where the integration constant is assumed to be zero. Substituting (11) into (8), we get

$$\Phi(\varphi) = -\frac{3}{k}\alpha\varphi^2 + \frac{2(k+3)}{k(k+2)}\beta\varphi - \frac{k}{4\pi G(k+1)}\alpha. \tag{12}$$

Substituting (10) and (12) into (7), we get

$$\left[\frac{\beta}{k+2}\varphi+\frac{k\alpha}{4\pi G(k+1)}\right]\Big/\left[\frac{\beta}{k+2}\varphi_0+\frac{k\alpha}{4\pi G(k+1)}\right]=\exp\left(\frac{\beta}{k+2}t\right), \tag{13}$$

where φ_0 is the value of scalar field when $t=0$. From (6), (10) and (12), we have

$$\begin{aligned} V(\varphi) &= \frac{\lambda}{4}\varphi^4-\frac{3\alpha\beta(k+3)}{k(k+2)}\varphi^3+\left[\frac{3\alpha^2(k-1)}{8\pi G(k+1)}+\frac{\beta^2(k+3)(3k+8)}{2k(k+2)^2}\right]\varphi^2 \\ &\quad -\frac{\alpha\beta(3k^2+7k)}{4\pi G(k+1)(k+2)}\varphi+\frac{3\beta^2}{8\pi G}-\frac{1}{2}\left[\frac{k\alpha}{4\pi G(k+1)}\right]^2, \end{aligned} \tag{14}$$

where the quartic coupling constant

$$\lambda = 6\alpha^2/k. \tag{15}$$

Finally, according to $H=\dot{a}/a$ and (10), (13), we find

$$\begin{aligned} a = a_0\exp\Bigg\{&\left[\frac{\alpha^2 k(k+2)}{\beta^2 4\pi G(k+1)}+\beta\right]t \\ &+\left[\frac{\alpha}{\beta}(k+2)\varphi_0+\frac{\alpha^2 k(k+2)^2}{\beta^2 4\pi G(k+1)}\right] \\ &\times\left[1-\exp\left(\frac{\beta}{k+2}t\right)\right]\Bigg\}, \end{aligned} \tag{16}$$

where a_0 is a positive constant. The (10) and (13)–(16) give an exact inflationary solution of the universe. From these equations we find that all physical variables are finite and analytic as t approaches to zero. Thus our solution can be free of initial singularity problem.

3 Scalar Density Fluctuation and Quartic Coupling Constant

In this section we will see whether our solution is consistent with the observed data of the universe or not. An accurate calculation of the perturbation spectra is in terms of the Hubble parameter following the conformal transformation $\hat{H}(\phi)$ and its derivatives, where ϕ is a new scalar field defined by

$$\frac{d\phi}{d\varphi}\equiv\frac{1}{\sqrt{8\pi G}}\sqrt{\frac{2f(\varphi)+3[f'(\varphi)]^2}{2f^2(\varphi)}}\approx\Omega^{-1}\sqrt{1+6\xi}, \tag{17}$$

where

$$\Omega^2 = 1+8\pi G_N\xi\varphi^2, \tag{18}$$

and $f(\varphi)\equiv(1+8\pi G\xi\varphi^2)/8\pi G$, and the strong coupling condition $8\pi G\xi\varphi^2\gg 1$ is assumed [18]. The metric of the conformal space-time is

$$d\hat{s}^2=\Omega^2[dt^2-a^2(t)dx^2]. \tag{19}$$

We make a particular conformal transformation [19]

$$\hat{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}, \tag{20}$$

and we further make the following change of variables

$$d\hat{t} = \Omega dt, \qquad \hat{a} = \Omega a, \tag{21}$$

then (19) can be translated into the same form as (2)

$$d\hat{s}^2 = d\hat{t}^2 - \hat{a}^2(\hat{t}) d\hat{x}^2. \tag{22}$$

Now we rewrite the theory in terms of these new variables as a minimally coupled theory with a metric (22) and an action

$$S = \int d^4\hat{x}\sqrt{-\hat{g}}\left[\frac{1}{16\pi G_N}\hat{R} + \frac{1}{2}\hat{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \hat{V}(\phi)\right], \tag{23}$$

where $\hat{V}(\phi) = \Omega^{-4} V(\varphi)$.

In the new theory, the spectral index n_s only depends on three parameters, which defined by

$$\varepsilon = \frac{1}{4\pi G_N}\left[\frac{\hat{H}'(\phi)}{\hat{H}(\phi)}\right]^2, \tag{24}$$

$$\eta = \frac{1}{4\pi G_N}\left[\frac{\hat{H}''(\phi)}{\hat{H}(\phi)}\right], \tag{25}$$

$$\zeta = \frac{1}{4\pi G_N}\left[\frac{\hat{H}'(\phi)\hat{H}'''(\phi)}{\hat{H}^2(\phi)}\right]^{1/2}, \tag{26}$$

where $\hat{H} = d(\ln \hat{a})/d\hat{t}$, and the primes denote the differentiation with respect to ϕ. When inflation ends, the parameter $\varepsilon(\phi_{end})$ is taken to the unity,

$$\varepsilon(\phi_{end}) = \frac{1}{4\pi G_N}\left[\frac{\hat{H}'(\phi_{end})}{\hat{H}(\phi_{end})}\right]^2 = 1. \tag{27}$$

To solve the puzzles of big bang fully, the scale factor should achieve enough expand during the inflation. Usually, the time of the last horizon crossing t_{Hc} is difficult to solve exactly, but should occur around 60 e-folds before the end of inflation, namely

$$\frac{\hat{a}(\hat{t}_{end})}{\hat{a}(\hat{t}_{Hc})} = \frac{\Omega(\varphi_{end})a(t_{end})}{\Omega(\varphi_{Hc})a(t_{Hc})} \approx e^{60}. \tag{28}$$

From (27), we can get the value of φ_{end}, and from (28) we can find the relationship between φ_{Hc} and φ_{end}, and then we can get the values of these parameters at the time of t_{Hc}. When $\alpha/\beta = 1$, $8\pi G$ is normalized to 1, and $\xi = 10^3$ [16], we get

$$\varepsilon(\phi_{Hc}) \approx 6.045 \times 10^{-4}, \tag{29}$$

$$\eta(\phi_{Hc}) \approx -0.0142, \tag{30}$$

$$\zeta(\phi_{Hc}) \approx -0.0142. \tag{31}$$

Clearly $\varepsilon(\phi_{Hc}) \ll |\eta(\phi_{Hc})|$, the spectral index of density fluctuations n_s beyond the slow-roll approximation, which is given by [20]

$$n_s = 1 - 4\varepsilon + 2\eta - 8(1+c)\varepsilon^2 + 2(3+5c)\varepsilon\eta - 2c\varepsilon\zeta \approx 0.972. \tag{32}$$

Using WMAP data only, the best fit value for n_s for the power-law flat ΛCDM model is $n_s = 0.958 \pm 0.016$ [21]. The result is consistent with it.

Quantum fluctuations of the scalar field produce density fluctuations on current astronomical scales when $\varphi \approx \varphi_{Hc}$. The fluctuations of density are amplified during inflation, which satisfy

$$\delta = \frac{\delta\rho}{\rho} = C\frac{\hat{H}^2(\phi)}{d\phi/d\hat{t}}, \tag{33}$$

where C is a model free constant. The result of COBE gives $\delta \approx 10^{-5}$. But the value is only an upper bound of (33) because the perturbations responsible for a large-scale structure can be formed by some other mechanisms. According to the result of COBE and (33) and (15) we find

$$\lambda \le 8.9 \times 10^{-11}, \tag{34}$$

where we have used $C = 1$, $\alpha/\beta = 1$, $8\pi G = 1$ and $\xi = 10^3$ [16]. This value is bigger than 10^{-13} reported previously [22].

4 Conclusions

In this paper, we present an exact inflationary solution of the universe in the chaotic model to non-minimally coupled scalar field from a specific $H(\varphi)$. In the course of calculating the spectral index n_s of density fluctuations, we find that when use any values of $\xi \gg 1$ and $\alpha/\beta \approx 1$ the results are consistent with the observed date and seldom varies, which implies a good applicability of our model. Furthermore, the value of λ is always bigger than 10^{-13} for any ξ, it shows that the non-minimal coupling can relax the constraint to quartic coupling constant, which agrees the idea of Fakir and Unruh [16].

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